

Frame Definability in Conditional Logic

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Abstract

In this paper we investigate classes of finite partially ordered sets that are definable by non-nested formulas in conditional logic. We discuss examples of such definable classes and introduce the notion of a c -morphism between posets as a tool to show that a class of finite posets is not definable. Using an analogue of the Jankov-Fine formulas from modal logic, we show that a class of finite posets is definable by a set of formulas if and only if it is closed under c -morphic images. Lastly, we prove a Sahlqvist-like correspondence theorem stating that every class of finite posets that is definable by a formula without nested conditionals is also definable by a first-order formula.

Keywords: conditional logic, frame definability, non-monotonic logic, belief revision, semiorers

1 Introduction

Conditional logic is a non-normal modal logic that extends propositional logic with a binary modality \rightsquigarrow that is called the (counterfactual) conditional. The guiding semantic intuition is that a conditional $\varphi \rightsquigarrow \psi$ is true if its consequent ψ is true at all worlds that are either most preferred, most plausible or maximally similar to the actual world among all the worlds that make the antecedent φ true. This intuition can be made precise considering an order over the set of worlds and then defining $\varphi \rightsquigarrow \psi$ to be true if ψ is true in all the minimal φ -worlds [13,17]. The same semantic clause is also used in other settings that are closely related to conditional logic, such as in default reasoning [28,15] and in belief revision theory [12,24].

The set of validities of conditional logic depends on what class of orders the semantics is based on. Lewis' conditional logic from [17] consists of all formulas that are valid with the above semantic clause over the class of all models based

on weak orders. A generalization of this logic was obtained later by Burgess [5] and Veltman [31], who axiomatize the validities over the class of all partial orders. In the literature on default reasoning and belief revision theory the validities of further classes of posets have been investigated, such as the class of interval orders [16,19] or the class of semiorders [22,25].

In this paper we systematically investigate the relation between validity in conditional logic and classes of finite posets. To this aim we adapt ideas from frame correspondence theory for normal modal logics [3, ch. 3] to conditional logic. The central notion for this paper is that of a class of finite posets being definable by a set of formulas in conditional logic. A class of posets is definable by a set of formulas if the class contains precisely the posets over which all the formulas in the set are valid. As an example one has that Lewis' conditional logic defines the class of weak orders.

The main technical contributions of this paper are as follows:

- (i) We provide a formula in conditional logic that defines the class of semiorders. To our knowledge this is a novel result. Characterizations of semiorders have been given in the context of choice functions [14,8] and belief revision theory [22,25]. However, these characterizations are formulated in the metalanguage and it is not clear how to express them in conditional logic.
- (ii) We characterize the definable classes of finite posets as those that are closed under c -morphic images. This result is similar to characterizations in modal logic, which state, roughly, that a class of frames is definable by modal formulas iff it is closed under generated subframes, coproducts and bounded morphic images [10]. As a consequence of our result we get that if a class of finite posets is not definable in conditional logic then there is a concrete counterexample of a c -morphism from a poset that is in the class to a poset that is not in the class.
- (iii) We provide a procedure that, given a formula in conditional logic, computes a first-order formula that is true in exactly those posets where the conditional formula is valid. This result can be seen as a simple version of the Sahlqvist Theorem for conditional logic.

The statement of the second result makes use of the notion of a c -morphism. This notion is inspired by the notion of a bisimulation between preferential models that was studied in the context of default reasoning by Zhu in [32]. A c -morphism in the sense defined in this paper is a function whose graph is a bisimulation in the sense of [32]. The precise formulation of the conditions in the definition of a c -morphism is quite technical. One part of the definition is the familiar back-condition from the definition of a bounded morphism; however, in general c -morphisms are not order-preserving. The notion of a c -morphism plays a role that is comparable to the notion of a bounded morphism, also called p -morphism, in modal logic. In particular it holds that any two models that are connected by a c -morphism satisfy the same formulas.

To our knowledge this paper is the first study of frame definability in con-

ditional logic. As such our approach still has the following limitations:

- (i) We only consider formulas of conditional logic in which the conditional is not nested and all occurrences of propositional letters are in the scope of the conditional. This allows us to focus on the aspects of definability that are specific to the minimization semantics of conditionals.
- (ii) We only consider finite posets. The main reason for this is that over non-wellfounded posets minimization does not yield a well-behaved conditional logic.
- (iii) We only consider frame definability relative to posets, that are transitive, reflexive and anti-symmetric relations. If one gives up anti-symmetry one obtains a semantics of the condition over preorders, that is reflexive and transitive relations. However, it has been shown that the logic of the class of preorders is the same as the logic of the more restricted class of posets [5,31]. If one additionally also removes the assumptions of transitivity or reflexivity then one obtains conditional logics that are weaker than the logic on posets. Following Hansson [13], such logics were studied mainly in the context of deontic logic [23,20,9,21], but more recently they have attracted broader attention [11,6]. We did not manage to adapt our techniques, especially the central notion of a *c*-morphism, to such settings.

The structure of this paper is as follows: In Section 2 we discuss the syntax and semantics of conditional logic and the notions of validity and definable classes. In Section 3 we provide examples of definable classes of posets that have arisen in the literature. In Section 4 we introduce the notion of a *c*-morphism, which we then use in Section 5 to prove for some examples of classes of posets that they are not definable. Section 6 contains the proof of the characterization result that a class of finite posets is definable iff it is closed under *c*-morphic images. In Section 7 we show that there is a first-order correspondent for every non-nested formula of conditional logic.

2 Preliminaries

In this section we discuss the language of conditional logic and its semantics over posets. We also define the notion of a definable class of posets.

2.1 Syntax

Conditional logics are commonly formulated in a classical propositional modal language with one binary modality \rightsquigarrow . A formula of the form $\varphi \rightsquigarrow \psi$ is called a *conditional* with *antecedent* φ and *consequent* ψ . In conditional logic conditional can be nested within the scope of other conditionals, as for example in the formula $((p \rightsquigarrow q) \rightsquigarrow r) \wedge q \rightarrow r$. In this paper we, however, only consider formulas in which the conditional is not nested and all propositional letters are in the scope of a conditional. To make this precise fix an infinite set \mathbf{Prop} of

propositional letters and consider the grammar:

$$\begin{aligned} \varphi_0 &::= p \mid \top \mid \neg\varphi_0 \mid \varphi_0 \wedge \varphi_0, & \text{where } p \in \text{Prop}, \\ \varphi_1 &::= \varphi_0 \rightsquigarrow \varphi_0 \mid \top \mid \neg\varphi_1 \mid \varphi_1 \wedge \varphi_1. \end{aligned}$$

Let \mathcal{L}_0 be the set of formulas generated from φ_0 and \mathcal{L}_1 the set of formulas generated from φ_1 . Note that \mathcal{L}_0 is just the language of classical propositional logic. In both \mathcal{L}_0 and \mathcal{L}_1 we use further Boolean connectives, such as \perp , \vee , \rightarrow , and \leftrightarrow , as abbreviations with their usual meaning in classical logic. To omit parenthesis we assume that \neg binds stronger than \wedge and \vee , which in turn bind stronger than \rightsquigarrow , \rightarrow and \leftrightarrow .

We are going to focus on formulas from \mathcal{L}_1 that are of the shape

$$\bigwedge_{i=1}^n (\varphi_i \rightsquigarrow \psi_i) \rightarrow \bigvee_{j=1}^m (\gamma_j \rightsquigarrow \delta_j),$$

where $\varphi_i, \psi_i, \gamma_j, \delta_j \in \mathcal{L}_0$ for all i and j . We call such formulas *inference rules* or simply *inferences* or *rules* and suggestively write them as

$$\frac{\varphi_1 \rightsquigarrow \psi_1 \quad \dots \quad \varphi_n \rightsquigarrow \psi_n}{\gamma_1 \rightsquigarrow \delta_1 \quad \dots \quad \gamma_m \rightsquigarrow \delta_m},$$

or as Σ/Γ , where $\Sigma = \{\varphi_i \rightsquigarrow \psi_i \mid 1 \leq i \leq n\}$ and $\Gamma = \{\gamma_j \rightsquigarrow \delta_j \mid 1 \leq j \leq m\}$. The elements of Σ are called the *premises* of the inference Σ/Γ and the elements of Γ are its *conclusions*. We allow for the cases where Σ is empty, meaning that the inference corresponds to a formula of the form $\top \rightarrow \bigvee \Gamma$, and where Γ is empty, meaning that the inference corresponds to the formula $\bigwedge \Sigma \rightarrow \perp$.

It is a consequence of Corollary 2.6 below, that for the purpose of understanding classes of posets that are definable by a formula in \mathcal{L}_1 it suffices to only consider formulas that are in the shape of inference rules. Focusing on the presentation of formulas as inference rules also matches the presentation in the setting of non-monotonic consequence relations, where such rules between conditional, thought of as non-monotonic inference relations, are taken as basic [15]. In Section 3 we provide multiple natural examples of such inference rules that have been discussed in the literature.

2.2 Semantics

The semantics of the conditional in conditional logic can be given in terms of ternary similarity relations \leq where $u \leq_w v$ holds if u is at least as similar to w as v [17, sec. 2.3]. A conditional then holds with respect to such a relation if the consequent is true at all the worlds that satisfy the antecedent and are maximally similar to the actual world among the worlds that satisfy the antecedent. In our setting, where we do not allow for the nesting of conditionals, one can however omit the relativization to the actual world and just minimize relative to a poset. This leads to the notion of a preferential model, which is the kind of semantic structure that is commonly used for non-monotonic consequence relations [15]:

Definition 2.1 Recall that a poset $P = (W, \leq)$ is a set W together with a partial order \leq on W , where a partial order is just a reflexive, transitive and anti-symmetric relation $\leq \subseteq W \times W$. Here, anti-symmetry denotes the property that whenever $w \leq v$ and $v \leq w$ hold for some $w, v \in W$ then it follows that $w = v$. A *preferential model* $M = (W, \leq, V)$ is a poset (W, \leq) together with a function $V : \text{Prop} \rightarrow \mathcal{P}W$. The elements of W are called *worlds* and the function V is called the *valuation function*.

In this paper we assume that all models are finite, meaning that the set of worlds W is a finite set. We use the notation $w < v$ as a shorthand for the conjunction of statements that $w \leq v$ holds and that $v \leq w$ does not hold.

The semantics of formulas from \mathcal{L}_0 and \mathcal{L}_1 is defined in the standard way. The set of worlds $\llbracket \varphi \rrbracket_V \subseteq W$, where a formula $\varphi \in \mathcal{L}_0$ is true, is computed by the recursive clauses $\llbracket p \rrbracket_V = V(p)$, $\llbracket \top \rrbracket_V = W$, $\llbracket \neg \varphi \rrbracket_V = W \setminus \llbracket \varphi \rrbracket_V$ and $\llbracket \varphi \wedge \psi \rrbracket_V = \llbracket \varphi \rrbracket_V \cap \llbracket \psi \rrbracket_V$. If V is clear, we write $\llbracket \varphi \rrbracket$ instead of $\llbracket \varphi \rrbracket_V$.

The semantic clauses for the propositional connectives over \mathcal{L}_1 relative to the model $M = (W, \leq, V)$ are

$$M \models \neg \varphi \text{ iff not } M \models \varphi, \text{ and } \quad M \models \varphi \wedge \psi \text{ iff } M \models \varphi \text{ and } M \models \psi.$$

For the semantics of the conditional we use the order \leq . A conditional is true if all of the minimal antecedent worlds satisfy the consequent:

$$M \models \varphi \rightsquigarrow \psi \text{ iff } \min_{\leq}(\llbracket \varphi \rrbracket) \subseteq \llbracket \psi \rrbracket.$$

The minimal worlds of a set $A \subseteq W$ in a partial order \leq over W can be defined as $\min_{\leq}(A) = \{x \in A \mid \forall y \in A (y \leq x \implies x \leq y)\}$.

Using a standard argument one can provide the following alternative formulation of the semantics for the conditional in finite orders:

Proposition 2.2 *Relative to all finite models (W, \leq, V) it holds that*

$$M \models \varphi \rightsquigarrow \psi \text{ iff for all } x \in \llbracket \varphi \rrbracket \text{ there is a } y \leq x \text{ with } y \in \llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket.$$

The notions of validity and of a definable class are then defined analogously to how they are defined for normal modal logics:

Definition 2.3 A formula $\varphi \in \mathcal{L}_1$ is *valid* in a poset $P = (W, \leq)$ if for all valuations $V : \text{Prop} \rightarrow \mathcal{P}W$ we have that $M \models \varphi$ for the model $M = (W, \leq, V)$. We can extend validity to sets of formulas and classes of posets: A set of formulas $\Sigma \subseteq \mathcal{L}_1$ is valid in a class of posets \mathcal{C} if φ is valid in P for all formulas $\varphi \in \Sigma$ and P in the class \mathcal{C} . If a formula, or set of formulas, is not valid in a poset, or class of posets, we also say that the formula, or set of formulas, is *falsifiable* in the poset, or class of posets and that the poset, or class of posets, *falsifies* the formula, or set of formulas.

Definition 2.4 A formula $\varphi \in \mathcal{L}_1$ *defines* a class of finite posets \mathcal{C} if and only if for all finite posets P it holds that

$$P \text{ is in } \mathcal{C} \text{ iff } \varphi \text{ is valid in } P.$$

Similarly, a set of formulas $\Sigma \subseteq \mathcal{L}_1$ *defines* a class of finite posets \mathcal{C} iff for all finite posets P it holds that P is in \mathcal{C} if and only if Σ is valid in P . A class of finite posets \mathcal{C} is *definable* if there is some set $\Sigma \subseteq \mathcal{L}_1$ such that Σ defines \mathcal{C} .

We conclude this section by showing that in order to study definable classes it suffices to consider formulas that are in the syntactic shape of inference rules.

Proposition 2.5 *Every formula $\varphi \in \mathcal{L}_1$ is equivalent to a conjunction of inference rules.*

Proof. This follows with propositional reasoning. To this aim consider $\varphi \in \mathcal{L}_1$ as a propositional formula where the conditionals are atoms. It is clear that we can rewrite φ into an equivalent conjunctive normal form, i.e, a conjunction of disjunctions of literals. It is also clear that every such disjunction of literals

$$\bigvee_{i=1}^n \neg(\varphi_i \rightsquigarrow \psi_i) \vee \bigvee_{j=1}^m (\gamma_j \rightsquigarrow \delta_j)$$

is propositionally equivalent to the inference

$$\bigwedge_{i=1}^n (\varphi_i \rightsquigarrow \psi_i) \rightarrow \bigvee_{j=1}^m (\gamma_j \rightsquigarrow \delta_j).$$

□

By observing that a conjunction is valid iff all of its conjuncts are valid we obtain the following Corollary:

Corollary 2.6 *For every formula $\varphi \in \mathcal{L}_1$ there is a finite set of inferences $\sigma = \{\Sigma_1/\Gamma_1, \dots, \Sigma_n/\Gamma_n\}$ such that σ is valid in a poset (W, \leq) if and only if φ is valid in (W, \leq) .*

3 Definable classes of posets

In this section we discuss examples of classes of finite posets that are definable by a formula in conditional logic. Figure 1 provides an overview of the examples from this section. The first-order formulas that describe the classes in the second column should be understood such that all free variables are universally quantified. The rules defining linear orders and orders with a minimum have an empty set of premises. Recall from Section 2.1 that we read the conclusion of these rules disjunctively.

Example 3.1 (Antichains) The class of antichains is defined by the rule

$$\frac{\top \rightsquigarrow p}{\neg p \rightsquigarrow \perp}.$$

In order to see this, suppose that $P = (W, \leq)$ is a poset which is not an antichain. Define the valuation V such that $V(p) = \min_{\leq}(W)$, thus p is true at all the minimal elements of P . Then $\top \rightsquigarrow p$ is true but we can show

Class	First-order formula	Defining rule
Antichains	$x \leq y \rightarrow y \leq x$	$\frac{\top \rightsquigarrow p}{\neg p \rightsquigarrow \perp}$
With minimum	$\exists x \forall y (x \leq y)$	$\overline{\top \rightsquigarrow p} \quad \overline{\top \rightsquigarrow \neg p}$
Linear orders	$x \leq y \vee y \leq x$	$\overline{p \rightsquigarrow q} \quad \overline{p \rightsquigarrow \neg q}$
Weak orders	$x < y \rightarrow z < y \vee x < z$	$\frac{p \rightsquigarrow q}{p \rightsquigarrow \neg r \quad p \wedge r \rightsquigarrow q}$
Interval orders	$x < y \wedge z < u \rightarrow x < u \vee z < y$	$\frac{p \vee r \rightsquigarrow q}{p \rightsquigarrow q \quad r \rightsquigarrow q}$
Semitransitive orders	$x < y \wedge y < z \rightarrow x < u \vee u < z$	$\frac{p \rightsquigarrow q \quad p \wedge \neg q \rightsquigarrow \neg r}{p \rightsquigarrow \neg r \quad p \wedge r \rightsquigarrow q}$

Fig. 1. Examples of definable classes of posets.

that $\neg p \rightsquigarrow \perp$ is false: because P is not an antichain, there is at least one element $y \in W$ that is not minimal, hence $\llbracket \neg p \rrbracket = W \setminus \min_{\leq}(W) \neq \emptyset$, and thus $\min_{\leq}(\llbracket \neg p \rrbracket) \not\subseteq \emptyset = \llbracket \perp \rrbracket$.

Viceversa, if an antichain $P = (W, \leq)$ satisfies $\top \rightsquigarrow p$ it means that all the minimal elements of P satisfy p . But since P is an antichain, all its elements are minimal and because $\llbracket p \rrbracket \cap \llbracket \neg p \rrbracket = \emptyset$ it must be $\llbracket \neg p \rrbracket = \emptyset$, therefore $\emptyset \subseteq \llbracket \perp \rrbracket$.

Example 3.2 (Orders with a minimum) It is easy to check that

$$\overline{\top \rightsquigarrow p} \quad \overline{\top \rightsquigarrow \neg p}$$

defines the class of all posets that have a unique minimal element.

Example 3.3 (Linear orders) We leave it to the reader to convince themselves that the rule

$$\overline{p \rightsquigarrow q} \quad \overline{p \rightsquigarrow \neg q}$$

defines the class of all linear orders. Note that this rule has no premises and thus expresses the formula $(p \rightsquigarrow q) \vee (p \rightsquigarrow \neg q)$. In [17, sec. 3.4] Lewis calls this formula “conditional excluded middle” and argues that its validity is characteristic of Stalnaker’s account of conditionals from [29].

Example 3.4 (Weak orders) The class of weak orders is defined by the rule

$$\frac{p \rightsquigarrow q}{p \rightsquigarrow \neg r \quad p \wedge r \rightsquigarrow q}.$$

Note that weak orders are also called strict weak orders and they can also be represented as total preorders. They provide a semantics for conditional

logic that is equivalent to Lewis' systems of spheres [17, sec. 2.3]. In belief revision theory they provide a semantics for the classic AGM postulates for belief revision [1,12]. In the guise of total preorders they are also behind the standard semantics of dynamic epistemic logic [2]. The inference rule used here to define weak orders is the rule of "rational monotonicity" from [16].

Example 3.5 (Interval orders) It is not hard to show that the following rule defines interval orders

$$\frac{p \vee r \rightsquigarrow q}{p \rightsquigarrow q \quad r \rightsquigarrow q}.$$

This rule has been called "disjunctive rationality" in the literature on non-monotonic reasoning [16,19,4]. Interval orders derive their name from the observation that they are precisely the orders that can be represented by a natural ordering of arbitrary length intervals on the line [7]. The interval order condition also plays a role in the semiorders that we discuss as the next example.

Example 3.6 (Semitransitive orders and semiorders) Semitransitive orders are defined by the inference rule

$$\frac{p \rightsquigarrow q \quad p \wedge \neg q \rightsquigarrow \neg r}{p \rightsquigarrow \neg r \quad p \wedge r \rightsquigarrow q}. \quad (1)$$

As semiorders are semitransitive interval orders, it follows that semiorders are defined by this rule together with the disjunctive rationality rule from Example 3.5. It has been argued that semiorders model human preference more adequately than weak orders [18,30]. It can be shown that they are representable, analogously to interval orders, by intervals of constant length [27]. Semiorders have been axiomatized in the context of choice functions [14,8] and in belief revision theory [22,25]. However, both settings use axioms that are not obviously expressible in the language of conditional logic.

To prove that the rule in (1) defines the class of semitransitive posets first assume that $P = (W, \leq)$ is a poset that fails to be semitransitive. This means that there are points x, y, z and u with $x < y < z$ such that u is incomparable to x and z . Thus, P has a subposet that looks as follows:

$$\begin{array}{ccc} z : p\bar{q}r & & \\ | & & \\ y : p\bar{q}\bar{r} & u : pqr & \\ | & & \\ x : p\bar{q}\bar{r} & & \end{array}$$

It is easy to see that in the model with a valuation V such that $V(p) = \{x, y, z, u\}$, $V(q) = \{x, u\}$ and $V(r) = \{z, u\}$ all the premises of the rule in (1) are true while all the conclusions are false.

For the other direction assume that the rule in (1) is valid in a poset $P = (W, \leq)$. To see that then P is semitransitive choose any points x, y, z and u in P such that $x < y < z$. To show that then either $x < u$ or $u < z$ consider a valuation V with $V(p) = \{x, y, z, u\}$, $V(q) = \{x, u\}$ and $V(r) = \{z, u\}$. It

is clear that this makes the conditionals $p \rightsquigarrow q$ and $p \wedge \neg q \rightsquigarrow \neg r$ true because $\min_{\leq}(\llbracket p \rrbracket) \subseteq \{x, u\} = \llbracket q \rrbracket$ and $\min_{\leq}(\llbracket p \wedge \neg q \rrbracket) = \min_{\leq}(\{y, z\}) = \{y\} \subseteq \llbracket \neg r \rrbracket$. It follows that either $p \rightsquigarrow \neg r$ or $p \wedge r \rightsquigarrow q$ is true in (W, \leq, V) . The former means that $\min_{\leq}(\{x, y, z, u\}) \subseteq \{x, y\}$, which entails $x < u$, and the latter means that $\min_{\leq}(\{z, u\}) \subseteq \{x, u\}$, which means that $u < z$.

4 C-morphisms

In this section we introduce c-morphisms, which we will use to show that a class of posets is not definable by a formula in conditional logic. C-morphisms are precisely those functions whose graph is a bisimulation in the sense of [32].

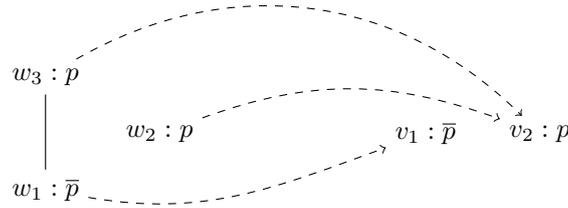
Definition 4.1 A *c-morphism* f from a poset $P = (W, \leq)$ to a poset $P' = (W', \leq')$ is a function $f : W \rightarrow W'$ such that:

- (i) For all $w \in W$ and $u' \leq' f(w)$ there is a $u \leq w$ such that $f(u) = u'$;
- (ii) For all $w' \in W'$ there is a $w \in W$ such that $f(w) = w'$ and for all $u \leq w$ we have that $f(u) \leq' w'$.

A poset P' is a *c-morphic image* of a poset P if there is some c-morphism from P to P' . We can extend the notion of a c-morphism to models such that $f : W \rightarrow W'$ is a c-morphism from $M = (W, \leq, V)$ to $M' = (W', \leq', V')$ if f is a c-morphism from (W, \leq) to (W', \leq') and $V(p) = f^{-1}(V'(p))$ for all $p \in \text{Prop}$.

Note that the first condition in the definition of c-morphisms is just the back condition for bounded morphisms in modal logic. Also note that it follows from the second condition that every c-morphism is surjective. However, the following examples shows that c-morphisms need not be order-preserving:

Example 4.2 The following is a picture of a c-morphism between two preferential models. The first models contains the worlds w_1 , w_2 and w_3 , and the second model contains the worlds v_1 and v_2 . The mapping of the c-morphism is depicted with the dashed arrows.



C-morphisms can be characterized via the preservation of minimal elements:

Proposition 4.3 Let $P = (W, \leq)$ and $P' = (W', \leq')$ be finite posets, $f : W \rightarrow W'$ is a c-morphism if and only if for every subset $X' \subseteq W'$ we have that $f(\min_{\leq}(f^{-1}(X'))) = \min_{\leq'}(X')$.

Proof. For the left-to-right direction, assume that $f : W \rightarrow W'$ is a c-morphism from P to P' . We prove that $f(\min_{\leq}(f^{-1}(X'))) = \min_{\leq'}(X')$ holds

for every $X' \subseteq W'$:

The \subseteq -inclusion: Let $w' \in f(\min_{\leq}(f^{-1}(X')))$. Hence, there must be $w \in \min_{\leq}(f^{-1}(X'))$ such that $f(w) = w'$. To prove that $w' \in \min_{\leq'}(X')$ it suffices to show that, for any $u' \in X'$ with $u' \leq' w'$, $u' = w'$. Thus, let u' be any such world. By condition (i) from the definition of c-morphisms we have that there is $u \leq w$ with $f(u) = u'$. Hence, $u \in f^{-1}(X')$. As $w \in \min_{\leq}(f^{-1}(X'))$ we have that $u = w$, hence $u' = w'$, proving the claim.

The \supseteq -inclusion: Let $w' \in \min_{\leq'}(X')$ and assume for a contradiction that there is no $w \in \min_{\leq}(f^{-1}(X'))$ such that $f(w) = w'$. We use an induction to construct an infinite descending chain $w_0 > w_1 > \dots$ in $f^{-1}(X')$, which contradicts the finiteness of W . By condition (ii) for c-morphisms there is some $w_0 \in W$ such that $f(w_0) = w'$ and for all $u \leq w_0$ we have that $f(u) \leq' w'$. Inductively, we then assume that we are given some $w_i \leq w_0$ such that $f(w_i) = w'$. Clearly, this inductive assumption is satisfied by w_0 in the base case. Since $f(w_i) = w'$ it follows by our assumption that $w_i \notin \min_{\leq}(f^{-1}(X'))$. This means that there is some $w_{i+1} < w_i$ such that $f(w_{i+1}) \in X'$. By instantiating the u from the condition on w_0 with w_{i+1} we get that $f(w_{i+1}) \leq w'$. Because $w' \in \min_{\leq'}(X')$ it follows from $f(w_{i+1}) \in X'$ and $f(w_{i+1}) \leq w'$ that $f(w_{i+1}) = w'$. Hence, w_{i+1} satisfies our inductive assumption as well.

For the right-to-left direction of the proposition, assume that f is such that for any $X' \subseteq W'$ we have that $f(\min_{\leq}(f^{-1}(X'))) = \min_{\leq'}(X')$. We show that f satisfies both conditions for c-morphisms:

Condition (i): Let $w \in W$ and $u' \leq' f(w) = w'$ be arbitrary. We show that for some $u \leq w$ we have that $f(u) = u'$. If $u' = w'$ then by reflexivity it trivially follows that $w \leq w$ and $f(w) = u'$. Hence, assume that $u' \neq w'$ and take $X' = \{u', w'\}$. By assumption we get that $f(\min_{\leq}(f^{-1}(X'))) = \min_{\leq'}(X') = \{u'\}$. Hence, as $w' \notin \min_{\leq'}(X')$, we have that $w \notin \min_{\leq}(f^{-1}(X'))$. Therefore, we deduce that there must be some $u \leq w$ with $u \in \min_{\leq}(f^{-1}(X'))$. As u is minimal in $f^{-1}(X')$, it follows that $f(u) = u'$, proving the claim.

Condition (ii): Let $w' \in W'$ be arbitrary and assume by way of contradiction that for all $w \in W$ with $f(w) = w'$ there is some $u \leq w$ such that $f(u) \not\leq' w'$. We construct an infinite chain $X'_0 \subsetneq X'_1 \subsetneq \dots$ of larger and larger subsets in W' , leading to a contradiction with the assumption that W' is finite. This chain is constructed by an induction, where at every step i we guarantee that w' is a minimal element of X'_i . In the base case we set $X'_0 = \{w'\}$, for which we obviously have that w' is a minimal element of the set. In the inductive step assume that we are given X'_i such that w' is a minimal element of w' . Because $f(\min_{\leq}(f^{-1}(X'_i))) = \min_{\leq'}(X'_i)$ there is then some $w \in \min_{\leq}(f^{-1}(X'_i))$ such that $f(w) = w'$. It follows that there is some $u \leq w$ such that $f(u) \not\leq' w'$. Define $X'_{i+1} = X'_i \cup \{f(u)\}$. Because $f(u) \not\leq' w'$ it holds again that w' is a minimal element of X'_{i+1} . To show that $X'_i \subsetneq X'_{i+1}$ we argue that $f(u) \notin X'_i$. Assume for contradiction that $f(u) \in X'_i$. Then, $u \in f^{-1}(X'_i)$. Because $w \in \min_{\leq}(f^{-1}(X'_i))$ and $u \leq w$ it would follow that $u = w$, which is impossible because $f(u) \not\leq' w' = f(w)$. \square

The crucial property of c-morphisms is that they preserve the truth of

formulas in conditional logic:

Proposition 4.4 *Let $M = (W, \leq, V)$ and $M' = (W', \leq', V')$ be finite preferential models and f a c-morphism from M to M' . Then, it holds for all formulas $\varphi \in \mathcal{L}_1$ that*

$$M \models \varphi \quad \text{iff} \quad M' \models \varphi.$$

Proof. The proof is an induction on the complexity of formulas $\varphi \in \mathcal{L}_1$.

First, note that it follows with an easy induction on the complexity of formulas $\varphi \in \mathcal{L}_0$ that $\llbracket \varphi \rrbracket_V = f^{-1}(\llbracket \varphi \rrbracket_{V'})$. The base case $V(p) = f^{-1}(V'(p))$ holds for all $p \in \mathbf{Prop}$ because f is a c-morphism of models.

The main inductive case is where $\varphi = \chi \rightsquigarrow \psi$ is a conditional, with $\chi, \psi \in \mathcal{L}_0$. First, observe that because the direct image map is left adjoint to the inverse image map we have that

$$f(\min_{\leq}(\llbracket \varphi \rrbracket_V)) \subseteq \llbracket \psi \rrbracket_{V'} \quad \text{iff} \quad \min_{\leq}(\llbracket \varphi \rrbracket_V) \subseteq f^{-1}(\llbracket \psi \rrbracket_{V'}). \quad (2)$$

Because from the base case for formulas in \mathcal{L}_0 we have that $f^{-1}(\llbracket \psi \rrbracket_{V'}) = \llbracket \psi \rrbracket_V$, it follows that the right side of (2) is equivalent to $\min_{\leq}(\llbracket \varphi \rrbracket_V) \subseteq \llbracket \psi \rrbracket_V$, which means that $M \models \varphi \rightsquigarrow \psi$. On the other hand using that f is a c-morphism we get from Proposition 4.3 that $f(\min_{\leq}(f^{-1}(\llbracket \varphi \rrbracket_{V'}))) = \min_{\leq'}(\llbracket \varphi \rrbracket_{V'})$. Combining it with $f^{-1}(\llbracket \psi \rrbracket_{V'}) = \llbracket \psi \rrbracket_V$, which follows from the base case, it implies that the left side of (2) is equivalent to $\min_{\leq'}(\llbracket \varphi \rrbracket_{V'}) \subseteq \llbracket \psi \rrbracket_V$. This is precisely the semantics of $M' \models \varphi \rightsquigarrow \psi$. The remaining cases for the propositional connectives in \mathcal{L}_1 follow by standard reasoning. \square

By using the preservation result we have just shown, we can also prove that c-morphisms preserve validities in posets:

Theorem 4.5 *Let f be a c-morphism from a finite poset $P = (W, \leq)$ to a finite poset $P' = (W', \leq')$. Then, every formula $\varphi \in \mathcal{L}_1$ that is valid in P is also valid in P' .*

Proof. We reason by contraposition: assume that φ is not valid in P' . Then, there is some valuation $V' : \mathbf{Prop} \rightarrow \mathcal{P}W'$ such that for the derived model $M' = (W', \leq', V')$ we have that $M' \not\models \varphi$. We lift V' to a valuation $V : \mathbf{Prop} \rightarrow \mathcal{P}W$ for P by exploiting f : for $p \in \mathbf{Prop}$, take $V(p) = f^{-1}(V'(p))$. Let $M = (W, \leq, V)$ be the derived model: by definition we now have that f is a c-morphism between the models M and M' . Hence, by Proposition 4.4, it follows that $M \not\models \varphi$, i.e. φ is not valid in P . \square

One might rephrase Theorem 4.5 as stating that definable classes of posets are closed under c-morphic images. Thus, this result can be used to show that some class of posets \mathcal{C} is not definable by a formula, or set of formulas, in conditional logic: if there are two posets P and P' such that P is in \mathcal{C} but P' is not in \mathcal{C} , and there is a c-morphism from P to P' , then \mathcal{C} is not definable.

5 Classes of posets that are not definable

In this section we will use Theorem 4.5 to give examples of classes of posets that are not definable by formulas in conditional logic.

We first make an observation about c-morphic images, which allows us to easily show for many classes of posets that they are not definable. The observation is that for every poset P with n minimal elements there is a c-morphism from P to the antichain with n elements A_n . Therefore, every class of posets that contains a poset with n minimal elements but does not contain A_n is not definable.

Lemma 5.1 *Let A_n be the antichain with n elements. Then, for every finite poset P with at least n minimal elements, we have that there is a c-morphism f from P to A_n .*

Proof. Let $P = (W, \leq)$ be a finite poset with an enumeration m_1, \dots, m_k of its k distinct minimal elements for $k \geq n$, and let a_1, \dots, a_n be an enumeration of the n elements in the antichain A_n . We define a surjective function g from $\{m_1, \dots, m_k\}$ to $\{a_1, \dots, a_n\}$ by setting $g(m_i) = a_i$ for $i \leq n$ and $g(m_i) = a_1$ for $i > n$. Then, observe that because P is finite we have for every $w \in W$ some minimal element m_w of P such that $m_w \leq w$. We then define the c-morphism f by mapping each $w \in W$ to the element $g(m_w)$ of A_n .

The definition of f satisfies condition (i) of Definition 4.1 because for every $w \in W$ and $u' \leq f(w)$ we have that $f(w) = u'$, as A_n is an antichain.

For condition (ii) note that for every $w' \in W$ we have $w' = f(m)$ for some minimal element m of P . If we then consider any $u \leq m$ we get that $u = m$, by the minimality of m , and hence it trivially holds that $f(u) = w'$. \square

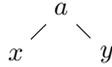
Together with Theorem 4.5 we obtain the following corollary:

Corollary 5.2 *If a class of finite posets contains a poset P with k minimal elements but does not contain the antichain A_n for some $n \leq k$ then the class is not definable.*

From this corollary we immediately obtain the following examples of classes that are not definable by formulas in conditional logic:

Example 5.3 The classes of posets with more than n elements for $n \geq 1$, with exactly n elements for $n > 1$, with more than n minimal/maximal elements for $n \geq 1$, and with chains longer than n for $n \geq 1$ are all not definable. In each of these classes we have some poset with at least 1 minimal element but all of these classes do not contain the antichain A_1 with exactly one element.

Example 5.4 The classes of connected posets, of posets with a maximum, and of join-semilattices, are not definable. All these classes contain the poset



with exactly 2 minimal and 1 maximal element. But all of these classes do not contain the antichain A_2 with 2 elements.

Lastly, we provide an example of an undefinable class for which we could not use the above corollary to prove its undefinability:

Example 5.5 The class of finite meet-semilattices is not definable. Consider the following two posets:



Clearly, the left poset is a meet-semilattice, while the right one is not. One can verify that the function f with $f(x) = 4'$ and $f(n) = n'$ for $n \in \{1, 2, \dots, 6\}$ defines a c-morphism from the left poset to the right one. To check condition (ii) of Definition 4.1 observe that for the element $4'$ on the right we have the element 4 on the left such that for each $n \leq 4$ it holds that $n' \leq 4'$.

6 The characterization theorem

We have already seen in Section 4 that every definable class of finite posets is closed under c-morphic images. In this Section we show that the converse is also true: every class of finite posets that is closed under c-morphic images is definable by a set of formulas in conditional logic. Thus, the definable classes of frames are precisely those that are closed under c-morphic images:

Theorem 6.1 *A class of finite posets \mathcal{C} is definable by some set of formulas $\Sigma \subseteq \mathcal{L}_1$ if and only if \mathcal{C} is closed under c-morphic images.*

To prove Theorem 6.1 we define an analogue of the Jankov-Fine formulas that can be used to show a similar characterization result for modal logic over finite transitive frames [3, sec. 3.4]. Thus, we define for every finite poset P a characteristic formula χ_P with the property that for every finite poset P' we get that P' falsifies χ_P iff P is a c-morphic image of P' .

Definition 6.2 Fix a finite poset $P = (W, \leq)$ such that $W = \{w_1, \dots, w_n\}$. For every point $w_i \in W$ define the formula $\alpha_i \in \mathcal{L}_0$ such that

$$\alpha_i = \bigvee \{p_j \mid w_j \not\leq w_i\}.$$

We then define the *characteristic formula* $\chi_P \in \mathcal{L}_1$ of P as the rule

$$\frac{\neg \bigvee_{j=1}^n p_j \rightsquigarrow \perp \quad \{p_i \wedge p_j \rightsquigarrow \perp \mid i \neq j\} \quad \{p_i \vee p_j \rightsquigarrow p_i \mid w_i < w_j\}}{\{p_i \rightsquigarrow \perp \mid 1 \leq i \leq n\} \quad \{p_i \vee \alpha_i \rightsquigarrow \alpha_i \mid 1 \leq i \leq n\}}.$$

To gain some intuition about what it means that this formula is falsified in a poset, recall that if the formula is falsified in $P' = (W', \leq')$ then there is some valuation V' over W' such that in the resulting model $M' = (W', \leq', V')$ we have that all the conditional that are premises of χ_P are true in M' , while

all the conditionals that are conclusions of χ_P are false. First, observe that because of the truth of the premise $\neg \bigvee_{j=1}^n p_j \rightsquigarrow \perp$ and the truth of the premise $p_i \wedge p_j \rightsquigarrow \perp$, for $i \neq j$ the sets $\llbracket p_i \rrbracket_{V'}$, for $i \in \{1, \dots, n\}$, partition W' . Because of the falsity of the conclusion $p_i \rightsquigarrow \perp$ there is at least one world that makes p_i true for every $i \in \{1, \dots, n\}$. Moreover, we have that:

- The truth of the premise $p_i \vee p_j \rightsquigarrow p_i$ for $w_i \leq w_j$ means that every p_j -world in M' is above some p_i -world.
- The falsity of the conclusion $p_i \vee \alpha_i \rightsquigarrow \alpha_i$ entails that there is at least one p_i -world in M' that has no p_j -world for $w_j \not\leq w_i$ below it.

Lemma 6.3 *For every finite poset P , P falsifies χ_P .*

Proof. Let $P = (W, \leq)$ and $W = \{w_1, \dots, w_n\}$. Define the valuation V such that $V(p_i) = \{w_i\}$ and consider $M = (W, \leq, V)$. It is clear that then the sets $\llbracket p_i \rrbracket_V$ partition W and thus the first two premises of χ_P are true in M and the first conclusion of χ_P is false. Also observe that if $w_i < w_j$ then $\min_{\leq}(\{w_i, w_j\}) = \{w_i\}$. Thus M makes the premise $p_i \vee p_j \rightsquigarrow p_i$ true. To see that M makes the conclusion $p_i \vee \alpha_i \rightsquigarrow \alpha_i$ false, for every fixed i , we use the alternative semantic clause from Proposition 2.2. Hence, we need to find some word $w \in \llbracket p_i \vee \alpha_i \rrbracket$ such that $v \not\leq w$ for all $v \in \llbracket \alpha_i \rrbracket$. Clearly we can take $w = w_i$ because every $v \in \llbracket \alpha_i \rrbracket$ is of the form $v = w_j$ for some $w_j \not\leq w_i$. \square

Proposition 6.4 *For every finite poset P' , P' falsifies χ_P if and only if there exists a c -morphism f from P' to P .*

Proof. The direction from right to left follows immediately from Lemma 6.3 together with Theorem 4.5.

For the direction from left to right, assume that $P' = (W', \leq')$ falsifies χ_P of some poset $P = (W, \leq)$ with $W = \{w_1, \dots, w_n\}$. Let V' be the valuation such that $M' \not\models \chi_P$ for $M' = (P', \leq', V')$. As already described above, note that this means, by the truth of the first premises, that the $\llbracket p_i \rrbracket_{V'}$ partition W' , and, by the falsity of the first conclusions, that none of the $\llbracket p_i \rrbracket_{V'}$ is empty. We then define a function $f : W' \rightarrow W$ such that $f(w') = w_i$ for the unique i such that $w' \in \llbracket p_i \rrbracket_{V'}$. We check that f is a c -morphism according to the conditions of Definition 4.1.

Condition (i): We need to show that for every $w' \in W'$ and $u \leq f(w')$ there is some $u' \leq' w'$ such that $f(u') = u$. Fix such w' and u and let $w_i = u$ and $w_j = f(w')$. If $u = f(w')$, then we can just let $u' = w'$ and thus we can assume that $u < f(w')$, implying that $w_i < w_j$. As the premise $p_i \vee p_j \rightsquigarrow p_i$ is true in M' and we have $w' \in \llbracket p_j \rrbracket_{V'}$, it follows that there must be some $u' \in \llbracket p_i \rrbracket_{V'}$ with $u' \leq' w'$. Then, from $u' \in \llbracket p_i \rrbracket_{V'}$, it follows that $f(u') = w_i = u$ and hence we have found a suitable u' .

Condition (ii): We need to show that for every $w_i \in W$ there is some $w' \in W'$ such that $f(w') = w_i$ and for every $u' \leq' w'$ we get $f(u') \leq w_i$. Fix w_i . Because the conclusion $p_i \vee \alpha_i \rightsquigarrow \alpha_i$ is false in M' it follows from Proposition 2.2 that there is some $w' \in \llbracket p_i \rrbracket_{V'}$ such that for all $u' \leq' w'$ it holds that $u' \notin \llbracket \alpha_i \rrbracket$. As $w' \in \llbracket p_i \rrbracket_{V'}$ we get that $f(w') = w_i$. Because α_i is defined

as $\bigvee \{p_j \mid w_j \not\leq w_i\}$ the claim that $u' \notin \llbracket \alpha_i \rrbracket$ for all $u' \leq' w'$ is equivalent to the claim that $f(u') \leq w_i$ holds for all $u' \leq w'$. \square

Proof. (Proof of Theorem 6.1) The direction from left to right follows from Theorem 4.5. For the direction from right to left, assume that \mathcal{C} is closed under c-morphic images. Then, let $\Gamma_{\mathcal{C}}$ be the set of all formulas valid on the class \mathcal{C} , that is, $\Gamma_{\mathcal{C}} = \{\varphi \in \mathcal{L}_1 \mid \varphi \text{ is valid in } P \text{ for all } P \text{ in } \mathcal{C}\}$. It is obvious from the definition all the formulas in $\Gamma_{\mathcal{C}}$ are valid in all the posets from \mathcal{C} . To show that $\Gamma_{\mathcal{C}}$ defines \mathcal{C} it suffices to show that if all of $\Gamma_{\mathcal{C}}$ is valid in some poset P , then $P \in \mathcal{C}$. Let P be such a poset. Consider its characteristic formula $\chi_P \in \mathcal{L}_1$. By Lemma 6.3 we know that P falsifies χ_P . Thus, it cannot be that $\chi_P \in \Gamma_{\mathcal{C}}$, which entails that there is some poset $P' \in \mathcal{C}$ that falsifies χ_P . From Proposition 6.4 it follows that P is a c-morphic image of P' and so P must also be in \mathcal{C} as the latter is closed under c-morphic images. \square

7 First-order correspondents

In this section we show that for every non-nested formula in conditional logic there is a formula in first-order logic that is true in precisely the finite posets where the formula of conditional logic is valid. Thus, every definable class of finite posets is elementary. For this result we are assuming a first-order language where the relations \leq and $<$ of posets and the equality relation are expressible. We then obtain the following:

Theorem 7.1 *For all $\sigma \in \mathcal{L}_1$ one can compute a first-order formula φ_{σ} such that φ_{σ} is true in a finite poset P if and only if σ is valid in P .*

Remark 7.2 We do not expect that Theorem 7.1 still holds for formulas with nested conditionals. To consider an analogue of the theorem for the full language of conditional logic one would have to consider frames to be ternary relations \leq such that for each world $w \in W$ the restricted relation \leq_w is a poset. We conjecture that there are non-elementary classes of such frames that are definable in nested conditional logic. Applying the translation of the modal diamond $\diamond\varphi$ as the conditional $\neg(\varphi \rightsquigarrow \perp)$ one can probably adapt examples of non-elementary classes that are definable in modal logic [3, sec. 3.2].

To prove Theorem 7.1, first observe that because of Corollary 2.6 it suffices to consider the case where σ is an inference of the form $\sigma = \Sigma/\Gamma$. We are going to define a first-order formula φ_{σ} that is true in some finite poset P iff σ is falsifiable in P . For the statement of the theorem we then need to consider the formula $\neg\varphi_{\sigma}$.

Let \mathcal{A} the set of all propositional assignments to the propositional letters occurring in σ . It can be thought of as the set of all functions from these letters to the truth values 0 and 1. Note that \mathcal{A} is a finite set. For every propositional formula $\alpha \in \mathcal{L}_0$ we define $\llbracket \alpha \rrbracket \subseteq \mathcal{A}$ to be the set of all assignments at which α is true in the sense from classical propositional logic.

Define $\mathfrak{S} \subseteq \Gamma\Sigma^*$ to be the set of all sequences

$$(\gamma \rightsquigarrow \delta, \alpha_1 \rightsquigarrow \beta_1, \dots, \alpha_n \rightsquigarrow \beta_n)$$

such that:

- (i) $\gamma \rightsquigarrow \delta \in \Gamma$ is some conclusion from Γ ;
- (ii) $\alpha_j \rightsquigarrow \beta_j \in \Sigma$ is some premise from Σ for all $j \in \{1, \dots, n\}$;
- (iii) The Σ -part of the sequence does not contain repetitions. This means that $\alpha_j \rightsquigarrow \beta_j \neq \alpha_k \rightsquigarrow \beta_k$ for all $j, k \in \{1, \dots, n\}$.

The *length* of a sequence $u \in \mathfrak{S}$ of the form $(\gamma \rightsquigarrow \delta, \alpha_1 \rightsquigarrow \beta_1, \dots, \alpha_n \rightsquigarrow \beta_n)$ is the number n . We consider singleton sequences $(\gamma \rightsquigarrow \delta) \in \mathfrak{S}$, where $\gamma \rightsquigarrow \delta \in \Gamma$, to be of length 0. Note that because of the last clause the length of sequences in \mathfrak{S} is bounded by the number of elements in Σ . Thus \mathfrak{S} is a finite set.

Let $u = (\gamma \rightsquigarrow \delta, \alpha_1 \rightsquigarrow \beta_1, \dots, \alpha_n \rightsquigarrow \beta_n)$ be some sequence from \mathfrak{S} and let $\alpha \rightsquigarrow \beta \in \Sigma$ be such that $\alpha \rightsquigarrow \beta \neq \alpha_i \rightsquigarrow \beta_i$ for all $i \in \{1, \dots, n\}$. We then write $u \cdot \alpha \rightsquigarrow \beta$ to denote the sequence $(\gamma \rightsquigarrow \delta, \alpha_1 \rightsquigarrow \beta_1, \dots, \alpha_n \rightsquigarrow \beta_n, \alpha \rightsquigarrow \beta) \in \mathfrak{S}$.

We then consider partial functions S from the disjoint sum $\mathfrak{S} + \mathcal{P}\Gamma$ of \mathfrak{S} and the powerset of Γ to the set \mathcal{A} . For any such partial function S let $D_S \subseteq \mathfrak{S} + \mathcal{P}\Gamma$ be the part of the domain on which S is defined. We call a partial function S from $\mathfrak{S} + \mathcal{P}\Gamma$ to \mathcal{A} *coherent* if

- (i) For all $\gamma \rightsquigarrow \delta \in \Gamma$ we have $(\gamma \rightsquigarrow \delta) \in D_S$ and $S((\gamma \rightsquigarrow \delta)) \in \llbracket \gamma \wedge \neg \delta \rrbracket$. Here, we consider $(\gamma \rightsquigarrow \delta)$ as a sequence in \mathfrak{S} and explicitly distinguish it from the singleton set $\{\gamma \rightsquigarrow \delta\} \in \mathcal{P}\Gamma$;
- (ii) If $u = (\gamma \rightsquigarrow \delta, \alpha_1 \rightsquigarrow \beta_1, \dots, \alpha_n \rightsquigarrow \beta_n)$ and $u \in D_S$ then $S(u) \in \llbracket \alpha_n \wedge \beta_n \rrbracket$;
- (iii) If $u = (\gamma \rightsquigarrow \delta, \alpha_1 \rightsquigarrow \beta_1, \dots, \alpha_{n-1} \rightsquigarrow \beta_{n-1}, \alpha_n \rightsquigarrow \beta_n)$ and $u \in D_S$ then $S(u) \notin \llbracket \alpha_i \rrbracket$ for all $i \in \{1, \dots, n-1\}$;
- (iv) If $u \in \mathfrak{S} \cap D_S$ and $S(u) \in \llbracket \alpha \wedge \neg \beta \rrbracket$ for some $\alpha \rightsquigarrow \beta \in \Sigma$ then $u \cdot \alpha \rightsquigarrow \beta \in D_S$. Note that by the previous two conditions it is not possible that $\alpha \rightsquigarrow \beta$ is a premise that already occurs in u . Hence, $u \cdot \alpha \rightsquigarrow \beta$ is well-defined as an element of \mathfrak{S} ;
- (v) If $\Delta \in D_S$ for some $\Delta \subseteq \Gamma$ then $S(\Delta) \notin \llbracket \alpha \wedge \neg \beta \rrbracket$ for all $\alpha \rightsquigarrow \beta \in \Sigma$.

Let \mathfrak{C} be the set of all coherent partial functions from $\mathfrak{S} + \mathcal{P}\Gamma$ to \mathcal{A} . Note that \mathfrak{C} is finite.

We are working with a set of first-order variables $\{x_u \mid u \in \mathfrak{S} + \mathcal{P}\Gamma\}$.

The first-order formula φ_σ is defined such that it states the existence of some coherent partial function $S \in \mathfrak{C}$ and points in the poset that for each element of D_S :

$$\varphi_\sigma = \bigvee_{S \in \mathfrak{C}} \exists x_u \dots u, v \in D_S \dots \exists x_v (\kappa(S) \wedge \psi(S) \wedge \chi(S) \wedge \mu_1(S) \wedge \mu_2(S)).$$

We use the notation $\exists x_u \dots u, v \in D_S \dots \exists x_v$ to denote a chain of existential quantifiers that contains a quantifier for every variable x_u for $u \in D_S$. The points corresponding to the elements of D_S are further constrained by the formulas $\kappa(S)$, $\psi(S)$, $\chi(S)$, $\mu_1(S)$ and $\mu_2(S)$ that all depend on S and contain only free variables of the form x_u for $u \in D_S$.

The formula $\kappa(S)$ requires that any two variables that are interpreted as

the same point must also map to the same assignment under S :

$$\kappa(S) = \bigwedge_{\substack{u, v \in D_S \\ S(u) \neq S(v)}} x_u \neq x_v.$$

The formula $\psi(S)$ requires that $\Delta \in D_S$ for every $\Delta \subseteq \Gamma$ for which there is some point in W that is below all the $x_{(\zeta)}$ for $\zeta \in \Delta$. Moreover, in this case x_Δ must be below all those $x_{(\zeta)}$:

$$\psi(S) = \bigwedge_{\Delta \subseteq \Gamma} (\exists y (\bigwedge_{\zeta \in \Delta} y < x_{(\zeta)}) \rightarrow \psi'(\Delta, S)),$$

where

$$\psi'(\Delta, S) = \begin{cases} \perp, & \text{if } \Delta \notin D_S, \\ \bigwedge_{\zeta \in \Delta} x_\Delta < x_{(\zeta)}, & \text{if } \Delta \in D_S. \end{cases}$$

The formula $\chi(S)$ states that if a sequence v from D_S extends another sequence u from D_S then x_v is below x_u in the poset:

$$\chi(S) = \bigwedge_{\substack{u \in D_S \cap \mathfrak{C} \\ v = u \cdot \xi \in D_S}} x_v < x_u.$$

The formula $\mu_1(S)$ requires that $x_{(\zeta)}$ for every $\zeta = \gamma \rightsquigarrow \delta \in \Gamma$ is minimal among all those points that interpret sequences that map under S to an assignment in $\llbracket \gamma \rrbracket$:

$$\mu_1(S) = \bigwedge_{\substack{\zeta \in \Gamma, u \in D_S \\ S(u) \in \llbracket \gamma \rrbracket}} \neg x_u < x_{(\zeta)}.$$

The formula $\mu_2(S)$ requires that x_u for every extended sequence $v = v' \cdot \alpha \rightsquigarrow \beta$, where $\alpha \rightsquigarrow \beta \in \Sigma$, is minimal among all those points that map to an assignment in $\llbracket \alpha \rrbracket$:

$$\mu_2(S) = \bigwedge_{\substack{v = v' \cdot \alpha \rightsquigarrow \beta \in D_S \\ u \in D_S, S(u) \in \llbracket \alpha \rrbracket}} \neg x_u < x_v.$$

It is clear that φ_σ can be computed from the inference σ . The remaining two lemmas of this section show that φ_σ does indeed express the falsifiability of σ .

Lemma 7.3 *If σ is falsifiable in a poset (W, \leq) then $(W, \leq) \models \varphi_\sigma$.*

Proof. Let $V : \text{Prop} \rightarrow \mathcal{P}W$ be a valuation such that $M = (W, \leq, V)$ makes all the premises in Σ true and all the conclusion in Γ false.

To show that φ_σ is true in the poset (W, \leq) we define a coherent partial function $S \in \mathfrak{C}$ and for every $u \in D_S$ an interpretation $w_u \in W$ for the variable x_u . We use $a(w) \in \mathcal{A}$ for any $w \in W$ as a shorthand for the assignment

$$a(w)(p) = \begin{cases} 1, & \text{if } w \in V(p), \\ 0, & \text{if } w \notin V(p). \end{cases}$$

Note that with this definition $w \in \llbracket \varphi \rrbracket_V$ iff $a(w) \in \llbracket \varphi \rrbracket$ holds for all $\varphi \in \mathcal{L}_0$.

The definition of S and the selection of w_u , for $u \in D_S$, proceeds in three steps that extend the domain of S . In each of the steps our choice of the w_u is such that that $S(u) = a(w_u)$ for all $u \in D_S$.

- (i) We first define S on sequences of the form $u = (\gamma \rightsquigarrow \delta)$ for all $\gamma \rightsquigarrow \delta \in \Gamma$. Fix a conclusion $\gamma \rightsquigarrow \delta \in \Gamma$. Because $\gamma \rightsquigarrow \delta$ is false in M , it follows that there is some world w that is minimal in $\llbracket \gamma \rrbracket_V$ such that $w \notin \llbracket \delta \rrbracket_V$. We let $w_u = w$ be this world and define $S(u) = a(w)$. Note that this ensures that S satisfies condition (i) from the definition of coherent functions.
- (ii) In this step we inductively extend the definition of S to longer and longer sequences from \mathfrak{S} such that condition (iv) becomes satisfied. This definition is by induction on the length of sequences $u \in \mathfrak{S}$. In every step, where we add a sequence $u = v \cdot \alpha \rightsquigarrow \beta$, we ensure that
 - (a) S and u satisfy conditions (ii) and (iii) for S being coherent,
 - (b) $w_u \leq w_v$, and
 - (c) $w_u \in \min_{\leq}(\llbracket \alpha \rrbracket_V)$.

The base case consists simply of all the sequences of length 0 that were added in the previous step. In the inductive step assume that we have already added all required sequences of length n . Let $v \in D_S$ be a sequence of length n and assume that $S(v) \in \llbracket \alpha \wedge \neg \beta \rrbracket$ for some $\alpha \rightsquigarrow \beta \in \Sigma$. We are going to add the sequence $u = v \cdot \alpha \rightsquigarrow \beta$ to the definitional domain of S . Because $S(v) = a(w_v)$ it follows that $w_v \in \llbracket \alpha \wedge \neg \beta \rrbracket_V$. Since $M \models \alpha \rightsquigarrow \beta$ there must be some $w \in \min_{\llbracket \alpha \rrbracket_V}$ with $w \in \llbracket \beta \rrbracket_V$ and $w < w_v$. We set $w_u = w$ and $S(u) = a(w_u)$. This takes care of items (b) and (c) for u . This definition also satisfies condition (ii) for S being coherent because $S(u) \in \llbracket \alpha \wedge \beta \rrbracket$.

To check that u satisfies condition (iii) assume that $u' = v' \cdot \alpha' \rightsquigarrow \beta'$ is a proper initial segment of u . We need to show that $S(u) \notin \llbracket \alpha' \rrbracket$, or equivalently that $w_u \notin \llbracket \alpha' \rrbracket_V$. Inductively we can assume that all initial segments of v already satisfy item (b) from above. Thus, $w_v \leq w_{v'}$ and together with $w_u < w_v$ we obtain $w_u < w_{v'}$. Moreover, because u' satisfies item (c) we know that $w_{u'} \in \min_{\leq}(\llbracket \alpha' \rrbracket_V)$. Combining these facts we obtain that $w_u \notin \llbracket \alpha' \rrbracket_V$.

- (iii) Lastly, we consider any subset $\Delta \subseteq \Gamma$ such that there exists some $w'_\Delta \in W$ such that $w'_\Delta < w_{(\gamma \rightsquigarrow \delta)}$ for all $\gamma \rightsquigarrow \delta \in \Delta$. Fix such a Δ , define w_Δ to be any minimal element of (W, \leq) that is below w'_Δ and set $S(\Delta) = a(w_\Delta)$. To see that this definition satisfies condition (v) on coherent partial functions we need to see that $w_\Delta \notin \llbracket \alpha \wedge \neg \beta \rrbracket_V$ for all $\alpha \rightsquigarrow \beta \in \Sigma$. If this was not the case then we would have that w_Δ is minimal in $\llbracket \alpha \rrbracket_V$, as it is minimal in W , but $w_\Delta \notin \llbracket \beta \rrbracket_V$. This would contradict the assumption that the model M makes the premise $\alpha \rightsquigarrow \beta \in \Sigma$ true.

It is clear that the partial function S that is defined in this way is coherent. It remains to be seen that the disjunct of φ_σ that corresponds to S is true in (W, \leq) . To this aim we interpret the existential variable x_u as the element

$w_u \in W$ for all $u \in D_S$. Because we have that $S(u) = a(w_u)$ for all $u \in D_S$ it is guaranteed that $\kappa(S)$ is true with this assignment. In the third step of the construction of S we make sure that $\psi(S)$ is true in (W, \leq) . The formula $\chi(S)$ holds because of item (b) from the second step. In the first step we chose $w_{(\gamma \rightsquigarrow \delta)} \in \min_{\leq}(\llbracket \gamma \rrbracket_V)$ and hence $\mu_1(S)$ is true in (W, \leq) . Lastly, $\mu_2(S)$ holds because of item (c) from the second step. \square

Lemma 7.4 *If $(W, \leq) \models \varphi_\sigma$ then σ is falsifiable in the poset (W, \leq) .*

Proof. Assume that the first-order formula φ_σ is true in the poset (W, \leq) . This means that there is some coherent $S \in \mathfrak{C}$ such that $\kappa(S)$, $\psi(S)$, $\chi(S)$, $\mu_1(S)$ and $\mu_2(S)$ hold for some interpretation of the existential variables from $\{x_u \mid u \in D_S\}$ in (W, \leq) . For all $u \in D_S$ let w_u be the value of the variable x_u for which this is the case. Define $X \subseteq W$ to be the set $X = \{w_u \in W \mid u \in D_S\}$.

Note that because $\kappa(S)$ holds for this interpretation of the existential variables it follows that $S(u) = S(v)$, whenever $w_u = w_v$ for some $u, v \in D_S$. For this reason the following function is well-defined $s : X \rightarrow \mathcal{A}, w_u \mapsto S(u)$.

Our next goal is to define a function $f : W \rightarrow X$ from which we then define the valuation $V : \text{Prop} \rightarrow \mathcal{P}W$ by setting

$$V(p) = \{w \in W \mid s(f(w))(p) = 1\}.$$

To define the value of $f(w) \in X$ for some $w \in W$, we distinguish cases depending on how w is situated relative to the elements in X .

- (i) If there is some $v \in X$ such that $v \leq w$ then we let $f(w) = y$ for some chosen $y \in X$ that is maximal among all $z \in X$ with $z \leq w$. Because X is finite such a maximal y always exists.
- (ii) If there is no $v \in X$ such that $v \leq w$ then we consider the set $\Delta = \{\zeta \in \Gamma \mid w < w_{(\zeta)}\}$. Because $\psi(S)$ holds of our assignment of variables we have that $\Delta \in D_S$. Thus, we can set $f(w) = w_\Delta \in X$.

Note that because of the first clause f is the identity on all $w \in X \subseteq W$.

It remains to be proven that $M = (W, \leq, V)$ makes all conditionals in Σ true and all conditionals in Γ false.

Thus, consider any premise $\alpha \rightsquigarrow \beta \in \Sigma$. To show that $M \models \alpha \rightsquigarrow \beta$ we use the reformulation of the semantic clause from Proposition 2.2. To this aim take any $w \in \llbracket \alpha \rrbracket_V$. We need to find a $w' \leq w$ with $w' \in \llbracket \alpha \wedge \beta \rrbracket_V$. Distinguish cases depending on the definition of $f(w)$.

First consider the case where there is no $v \in X$ such that $v \leq w$. Then $f(w) = w_\Delta$ for some $\Delta \subseteq \Gamma$. Because of condition (v) of coherence it holds that $w_\Delta \notin \llbracket \alpha \wedge \neg \beta \rrbracket_V$. Note that the definition of V is such that w satisfies the same propositional letters as w_Δ because $f(w) = w_\Delta$. Thus $w \notin \llbracket \alpha \wedge \neg \beta \rrbracket_V$. Because $w \in \llbracket \alpha \rrbracket_V$ it follows that $w \in \llbracket \alpha \wedge \beta \rrbracket_V$ and we can take $w' = w$.

In the other case there is some $v \in X$ with $v \leq w$ then let $y \in X$ be such that $y = f(w)$ and $y \leq w$. Because $y \in X$ we have that $y = w_u$ for some $u \in \mathfrak{S} + \mathcal{P}\Gamma$. We distinguish further cases depending on whether $u \in \mathcal{P}\Gamma$ or $u \in \mathfrak{S}$. If $u \in \mathcal{P}\Gamma$ then $u = \Delta$ for some $\Delta \subseteq \Gamma$ and we can reason precisely as

in the previous case. In the other case we have that $u \in \mathfrak{S}$. From $f(w) = w_u$ and $w \in \llbracket \alpha \rrbracket_V$ it follows that $w_u \in \llbracket \alpha \rrbracket_V$, because w and w_u satisfy the same propositional letters under V . Note that we can assume that $w_u \in \llbracket \alpha \wedge \neg \beta \rrbracket_V$ because if $w_u \in \llbracket \alpha \wedge \beta \rrbracket_V$ then also $w \in \llbracket \alpha \wedge \beta \rrbracket_V$ and we can set $w' = w$. But $w_u \in \llbracket \alpha \wedge \neg \beta \rrbracket_V$ means that $S(u) = s(w_u) \in \llbracket \alpha \wedge \neg \beta \rrbracket$. By condition (iv) from the definition of coherency this entails that $v = u \cdot \alpha \rightsquigarrow \beta \in D_S$. From condition (ii) we get that $S(v) \in \llbracket \alpha \wedge \beta \rrbracket$ and thus $w_v \in \llbracket \alpha \wedge \beta \rrbracket_V$. Using that $\chi(S)$ is true in (W, \leq) we have that $w_v \leq w_u$. Using $w_u = y \leq w$ it follows that $w_v \leq w$. Thus we can take $w' = w_v$.

Lastly, we argue that the conclusions are false in M . Consider any conclusion $\gamma \rightsquigarrow \delta \in \Gamma$. By condition (i) on the coherent function S we have that $S(\gamma \rightsquigarrow \delta) \in \llbracket \gamma \wedge \neg \delta \rrbracket$. Thus $w_{(\gamma \rightsquigarrow \delta)} \in \llbracket \gamma \wedge \neg \delta \rrbracket_V$. We are going to show the claim that for all $w \in W$ with $w < w_{(\gamma \rightsquigarrow \delta)}$ we have $w \notin \llbracket \gamma \rrbracket_V$. From this it then follows by the alternative formulation of the semantics in Proposition 2.2 that $M \not\models \gamma \rightsquigarrow \delta$. To prove the claim, consider any $w < w_{(\gamma \rightsquigarrow \delta)}$. We distinguish cases depending on the clause defining $f(w)$.

If there is some $v \in X$ with $v \leq w$ then consider the $y \in X$ with $y \leq w$ such that $f(w) = y$. By transitivity it follows that $y \leq w_{(\gamma \rightsquigarrow \delta)}$ and hence we can use that $\mu_1(S)$ is true to derive that $y \notin \llbracket \gamma \rrbracket_V$. Because $f(w) = y$ we have that the valuation V is the same on w as on y and hence also $w \notin \llbracket \gamma \rrbracket_V$.

If there is no $v \in X$ with $v \leq w$ then $f(w) = w_\Delta$ for some $\Delta \subseteq \Gamma$ such that $\gamma' \rightsquigarrow \delta' \in \Delta$ whenever $w < w_{(\gamma' \rightsquigarrow \delta')}$. Because $w < w_{(\gamma \rightsquigarrow \delta)}$ this means that $\gamma \rightsquigarrow \delta \in \Gamma$. Because $\psi(S)$ is true in (W, \leq) it follows that $w_\Delta < w_{(\gamma \rightsquigarrow \delta)}$ and because $\mu_1(S)$ is true it follows that $w_\Delta \notin \llbracket \gamma \rrbracket_V$. Because $f(w) = w_\Delta$ we can conclude that $w \notin \llbracket \gamma \rrbracket_V$. \square

8 Conclusion

This paper provides results on frame definability in conditional logic. Definable classes of posets are characterized by being closed under c-morphic images and every definable class of posets is elementary.

An obvious direction for further research is to lift some of the limitations of the setting. First, one might be interested in studying definability of ternary relations by formulas in full conditional logic, where the conditional can occur nested. We expect that to obtain results in this direction one would have to combine ideas from this paper with ideas from the work on frame definability for normal modal logic. Second, one might try to generalize to the infinite case. We conjecture that most of our results generalize to wellfounded orders. For non-wellfounded orders, however, frame-definability seems to behave quite different than in the finite case. Third, one might try to adapt our approach to a setting that gives up some of the assumptions of anti-symmetry, transitivity or reflexivity that come from working with posets. We expect this to be quite challenging.

A further interesting open question is whether there are any general completeness results for conditional logics, similar to the Sahlqvist's completeness theorem for modal logic [26]. Many of the examples of formulas that we give

in Section 3 were taken from literature that proves completeness results for the logic that is axiomatized by these formulas. One might hope that there is a general completeness result that gives a syntactic characterization a some class of formulas and then shows that if one adds formulas from this class as an additional axiom to the logic of Burgess [5] and Veltman [31], then one obtains a logic that is complete for the class of posets that the formula defines.

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